STATIONARY SOLUTIONS TO THE EQUATIONS OF THE QUASI-LINEAR APPROXIMATION FOR A PLASMA WITH COLLISIONS

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One of us [1] has considered the equations of the quasi-linear approximation for a plasma with collisions, which, for a spatially homogeneous plasma without a magnetic field in the stationary case, are as follows:

$$\frac{d}{dp_i}\frac{k_ik_j}{k^2}\left\{\frac{1}{2}\frac{e_a^2|E|^2\nu_a}{(\omega-\mathbf{kv})^2+\nu_a^2}\frac{df_a}{dp_j}\right\}+S_a=0,$$

$$\varepsilon'=0,\quad \varepsilon''=0\ , \tag{0.1}$$

$$\varepsilon(\omega,\mathbf{k})=\varepsilon'+i\varepsilon''=1\ +$$

$$+\sum_a\frac{4\pi e_a^2n_a}{\omega k^2}\int\frac{\mathbf{kv}}{\omega-\mathbf{kv}+i\nu_a}\left(\mathbf{k}\frac{df_a}{d\mathbf{p}}\right)d\mathbf{p}\ , \tag{0.2}$$

in which $\varepsilon(\omega, k)$ is the dielectric constant of the plasma for longitudinal waves ($k \| E$), $f_a(p)$ is the distribution function for particles of type a, S_a is the collision integral (which is put in the Landau form for a Coulomb plasma without allowance for polarization), and ν_a is the reciprocal relaxation time due to collisions in the rapidly varying (pulsating) part of the distribution function. In general, ν_a is a function of velocity, and it can be expressed via the distribution function for the background under the conditions for which the equations of the quasi-linear approximation were derived. The ν_a for a slightly nonequilibrium plasma may be considered as some effective collisional frequency averaged over the momenta, which for the electrons makes a certain contribution to the imaginary part of the dielectric constant, where ν_a can be calculated [2, 3] if f_a is known.

An important point is that Eqs. (0.1) were derived for a monochromatic longitudinal wave, and so it is not a necessary condition to have a sufficiently large width for the wave packet, which condition is characteristic of the quasi-linear theory previously given. However, as in the quasi-linear theory, it is assumed that the oscillation amplitude is small (so that nonlinear interaction between waves can be neglected), and so

$$e_a^2|E|^2/m_aT_a\omega^2\ll 1\ , \tag{0.3}$$

while the dimensionless parameter $e_a^2|E|^2/m_aT_a\nu_a^2$ may be on the order of, or even much greater than, unity.

In §1 of this paper we use the equations for the energy and momentum to derive expressions for the electron drift velocity and the difference $T_e - T_i$ (electron and ion temperatures) for a two-component plasma in a given wave field on the assumption of Maxwellian background distribution functions for the electrons and ions. Expression (1.14) for the temperature difference extends to longitudinal waves a result obtained in [4] on the heating of electrons in a strong electric field.

Section 2 deals with the amplitudes of steady-state high-frequency Langmuir waves when the plasma contains a steady electron beam. We derive in §3 the steady-state electron distribution function for the one-dimensional case on the basis of a model collision integral, and this differs only slightly from the Maxwellian case in which the spatial dispersion is slight.

§1. Electron drift and heating in the electric field of a wave. We use the quasi-linear approximation for the balance of momentum and energy for particles of type a. We multiply the first equation in (0.1) by $\mathbf{p}n_a$ and $n_ap^2/2m_a$ and integrate with respect to momentum, which gives, respectively, for the momentum and energy

$$\frac{\mathbf{k}}{\omega}\frac{|E|^2}{8\pi}J_a(\omega,\mathbf{k})=n_a\int\mathbf{p}S_ad\mathbf{p}\ , \tag{1.1}$$

$$\frac{|E|^2}{8\pi}J_a(\omega,\mathbf{k})=n_a\int\frac{p^2}{2m_a}S_ad\mathbf{p}\ , \tag{1.2}$$

in which

$$J_a(\omega,\mathbf{k})=\frac{4\pi e_a^2n_a}{k^2}\int\frac{\nu_a\mathbf{kv}}{(\omega-\mathbf{kv})^2+\nu_a^2}\left(\mathbf{k}\frac{df_a}{d\mathbf{p}}\right)d\mathbf{p}\ , \tag{1.3}$$

characterizes the change in momentum and energy of component a as a result of interaction between the particles and the wave.

In deducing (1.1) we have used an expression for the rapidly varying current density of component a

$$\mathbf{j}_a^1=-ie_a^2n_a\int\frac{\mathbf{v}}{\omega-\mathbf{kv}+i\nu_a}\left(\mathbf{E}\frac{df_a}{d\mathbf{p}}\right)d\mathbf{p}\ ,$$

in addition to

$$\langle\rho_a^1\mathbf{E}\rangle=\mathbf{k}\omega^{-1}\langle\mathbf{j}_a^1\mathbf{E}\rangle\ ,$$

which, in the quasi-linear approximation, relates $\mathbf{j}_a^1$ to the rapidly varying charge density ρ_a^1, while $\langle\,\rangle$ denotes averaging over the fast pulsations.

Summation of (1.1) and (1.2) over all a gives

$$\sum_a\langle\rho_a^1\mathbf{E}\rangle=\frac{\mathbf{k}}{\omega}\sum_a\langle\mathbf{j}_a^1\mathbf{E}\rangle=0\ ,$$

which expresses the fact that the work done by the electric field is zero in the steady state, and which is equivalent to the third equation in (0.1). *

We assume that the steady-state distribution functions for all the components are Maxwellian in the zeroth approximation, but that each component can have its own temperature and velocity:

$$f_a=(\sqrt{\pi}v_a)^{-3}\exp\left\{-\frac{(\mathbf{v}-\mathbf{u}_a)^2}{v_a^2}\right\},\qquad v_a^2=\frac{2T_a}{m_a}\ . \tag{1.4}$$

Consider the integral in (1.3). We set the x-axis along $\mathbf{k}$, integrate with respect to $p_\perp$, and convert to the dimensionless variables

*The method of division into background and pulsations used in deriving (0.1) differs from that used in the quasi-linear theory, where the pulsation spectrum must be fairly broad.

$$x_a = \frac{\omega - ku_{ax}}{kv_a}, \quad y_a = \frac{\nu_a}{kv_a},$$

$$\eta_a = \frac{u_{ax}}{v_a}, \quad \xi = \frac{v_x - u_{ax}}{v_a}, \tag{1.5}$$

which gives

$$J_a(\omega, \mathbf{k}) = -\frac{2}{\sqrt{\pi}} \frac{\omega_a^2}{k^2 v_a^2} \nu_a \int\limits_{-\infty}^{+\infty} \frac{\xi(\xi + \eta_a) e^{-\xi^2}}{(x_a - \xi)^2 + y_a^2} d\xi, \tag{1.6}$$

in which ω_a is the Langmuir frequency of component a. It is readily shown that the latter integral is expressed via the complex error integral

$$w(z) = e^{-z^2}\left\{1 + \frac{2i}{\sqrt{\pi}} \int\limits_0^z e^{t^2} dt\right\}, \quad z = x + iy,$$

with the balance equations taking the form

$$-\frac{\mathbf{k}}{\omega} \frac{|E|^2}{4\pi} \frac{\omega_a^2}{k^2 v_a^2} \nu_a \psi(z_a) = n_a \int \mathbf{p} S_a d\mathbf{p}, \tag{1.7}$$

$$-\frac{|E|^2}{4\pi} \frac{\omega_a^2}{k^2 v_a^2} \nu_a \psi(z_a) = n_a \int \frac{p^2}{2m_a} S_a d\mathbf{p}, \tag{1.8}$$

$$\psi(z_a) = 1 + \sqrt{\pi}\left\{\frac{x_a\eta_a + x_a^2 - y_a^2}{y_a} \operatorname{Re} w(z_a) - \right.$$

$$\left. - (\eta_a + 2x_a) \operatorname{Im} w(z_a)\right\}. \tag{1.9}$$

The integrals appearing on the right-hand side here have been calculated [1, 3, 5] and can be put as

$$n_a \int \mathbf{p} S_a d\mathbf{p} = -n_a m_a \sum_b \frac{\mathbf{u}_a - \mathbf{u}_b}{\tau_{ab}^{(u)}}, \tag{1.10}$$

$$n_a \int \frac{p^2}{2m_a} S_a d\mathbf{p} = -n_a \sum_b \left\{\frac{3}{2} \frac{T_a - T_b}{\tau_{ab}^{(T)}} + \right.$$

$$\left. + m_a \frac{(u_a - u_b)(u_a v_b^2 + u_b v_a^2)}{\tau_{ab}^{(u)} (v_a^2 + v_b^2)}\right\} \tag{1.11}$$

in which $\tau_{ab}^{(u)}$ and $\tau_{ab}^{(T)}$ are the relaxation times for the directional velocities and temperatures, respectively, in the collision of charged particles of type a with particles of type b.

If there is no spatial dispersion ($k = 0$), the directional velocities of the electrons and ions in a two-component plasma are zero, and we get

$$T_e - T_i = \frac{e^2 |E|^2 \nu_e \tau_{ei}^{(T)}}{3m(\omega^2 + \nu_e^2)}, \tag{1.12}$$

which is precisely the relation for electron heating in a strong electric field, because $\omega < \omega_e$ $\nu_e \tau_{ei}^{(T)} = {}^1/_2 M/m$ [3, 4].

We take the asymptotic expansion of $\psi(z_a)$ for $|z_a| \gg 1$ and retain only the first terms in k to get from (1.7), (1.8), (1.10), and (1.11) the equations for the electron velocity and temperature in the case of slight spatial dispersion:

$$\mathbf{u}_e = \frac{\mathbf{k}}{\omega} \frac{e^2 |E|^2 \nu_e \tau_{ei}^{(u)}}{2m^2(\omega^2 + \nu_e^2)}, \tag{1.13}$$

$$T_e = T_i + \frac{e^2 |E|^2 \nu_e \tau_{ei}^{(T)}}{3m(\omega^2 + \nu_c^2)}\left\{1 + \frac{3}{2} k^2 v_e^2 \frac{3\omega^2 - \nu_e^2}{(\omega^2 + \nu_e^2)^2}\right\}. \tag{1.14}$$

The latter equation does not take into account the directional velocity of the electrons, because $\nu_e \tau_{ei}^{(u)} = 1$ [3], and (1.12) gives $u_e \ll v_e$. These expressions correspond to high-frequency Langmuir waves for which $\omega \gg kv_e$.

§2. Steady-state waves in a plasma. The spectrum and amplitude of these waves may be found from the second and third equations in (0.1). Substitution from (1.4) into the expression for the dielectric constant in (0.2) gives the following in terms of the complex error integral:

$$\varepsilon'(\omega, \mathbf{k}) = 1 + \sum_a \frac{2\omega_a^2}{k^2 v_a^2} \varphi(z_a),$$

$$\varepsilon''(\omega, \mathbf{k}) = \sum_a \frac{2\omega_a^2 \nu_a}{k^2 v_a^2 \omega} \psi(z_a), \tag{2.1}$$

$$\varphi(z_a) = 1 - \sqrt{\pi}\left\{\left(y_a + \frac{x_a y_a}{x_a + \eta_a}\right) \operatorname{Re} w(z_a) + \right.$$

$$\left. + \left(x_a - \frac{y_a^2}{x_a + \eta_a}\right) \operatorname{Im} w(z_a)\right\}. \tag{2.2}$$

The function $\psi(z_a)$ and the dimensionless variables z_a and η_a have the meanings of §1 [see (1.5) and (1.9)].

Consider an electron-ion plasma with a steady electron beam whose directional velocity substantially exceeds the thermal velocity of the plasma electrons. The linear theory indicates that increasing longitudinal high-frequency ($\omega \approx \omega_e$) Langmuir waves are excited, whose phase velocity $\omega/k \approx u_0$ (beam velocity) and whose increment for a cold beam is $(n_0/n_e)^{1/3}$ (the subscript 0 subsequently denotes quantities referring to the beam, which is considered as an additional component of the plasma). Consider the dispersion equation for such waves.

We use the asymptotic expansion of $\psi(z_a)$ for $|z_a| \gg 1$, i.e., we assume that

$$(\omega - \mathbf{k}\mathbf{u}_a)^2 + \nu_a^2 \gg k^2 v_a^2,$$

which gives from (2.1) that

$$\varepsilon' = 1 - \sum_a \frac{\omega_a^2}{(\omega - \mathbf{k}\mathbf{u}_a)^2 + \nu_a^2} \times$$

$$\times \left\{1 - \frac{2\nu_a^2 \mathbf{k}\mathbf{u}_a}{\omega[(\omega - \mathbf{k}\mathbf{u}_a)^2 + \nu_a^2]}\right\} = 0, \tag{2.3}$$

$$\varepsilon'' = $$

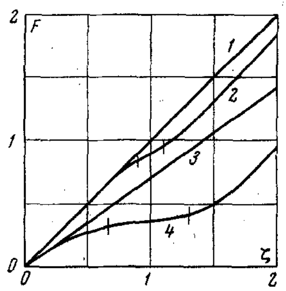
$$= \sum_a \frac{\nu_a \omega_a^2}{\omega[(\omega - \mathbf{k}\mathbf{u}_a)^2 + \nu_a^2]}\left\{1 + \frac{2\mathbf{k}\mathbf{u}_a(\omega - \mathbf{k}\mathbf{u}_a)}{(\omega - \mathbf{k}\mathbf{u}_a)^2 + \nu_a^2}\right\} = 0. \tag{2.4}$$

Passing to a collision-free plasma (ν_e, $\nu_0 \to 0$, $u_e \to 0$), we get from (2.3) the usual dispersion equation for the high-frequency Langmuir waves excited by a fast mono-energetic electron beam:

$$1 - \omega_e^2/\omega^2 - \omega_0^2/(\omega - \mathbf{k}\mathbf{u}_0)^2 = 0, \tag{2.5}$$

where the effect of the ions can be neglected in this case.

Now $\omega = \mathbf{k}\mathbf{u}_0 - \gamma$ in the linear theory, where γ is a small quantity proportional to the increment, and so it is reasonable to suppose that $(\omega - \mathbf{k}\mathbf{u}_0)^2 \gg \nu_0^2$, since for attainment of the quasi-linear state it is necessary



for the increment in the linear theory to exceed the damping decrement, which is proportional to ν_0. On this basis, and assuming also that

$$(\omega - \mathbf{k}\mathbf{u}_e)^2 \gg \mathrm{v}_e^2, \qquad \omega^2 \gg (\mathbf{k}\mathbf{u}_e)^2,$$

we put (2.4) in the form

$$\frac{\mathrm{v}_e \omega_e^2}{\omega^2}\left(1 + \frac{2\mathbf{k}\mathbf{u}_e}{\omega}\right) + \frac{2\mathrm{v}_0 \mathbf{k}\mathbf{u}_0 \omega_0^2}{(\omega - \mathbf{k}\mathbf{u}_0)^3} = 0\,, \tag{2.6}$$

whereupon (2.3) becomes (2.5).

This system may be solved by successive approximation. We assume that

$$\omega - \mathbf{k}\mathbf{u}_0 = \omega_e \gamma, \quad \gamma \ll 1\,. \tag{2.7}$$

Substitution of (2.7) and (2.5) into (2.6) gives the following when we neglect the terms $\gamma^{-2} n_0/n_e$ and $2\mathbf{k}\mathbf{u}_e/\omega$ relative to unity:

$$\gamma = -(\varkappa n_0/n_e)^{1/3}, \qquad \varkappa = 2\mathrm{v}_0/\mathrm{v}_e\,.$$

In the next approximation we incorporate the small terms in (2.5) and (2.6) in order to find u_e. Then we use the momentum-balance equation for the plasma electrons to get an expression for the field energy,

$$|E|^2/8\pi = {}^1\!/_2 m u_0^2 n_0 \gamma^{-2}(\varkappa + {}^1\!/_2)\,, \tag{2.8}$$

which implies that this energy substantially exceeds the kinetic energy of the beam, which is in qualitative agreement with the result [6] for the maximum energy of the oscillations in the injection of a monoenergetic beam into a semi-infinite plasma. All the same, the field energy of (2.8) remains much less than the thermal energy of the plasma electrons, in accordance with the existence of a small parameter in the quasi-linear theory of (0.3).

The low-frequency oscillations excited by a slow beam can similarly be examined in the quasi-linear approximation.

§3. Distribution function. The steady-state distribution function in the presence of longitudinal waves differs from the Maxwellian case and should be deduced from the first equation in (0.1), which in the general case is an integro-differential equation for f_a, while the distribution functions for the other components in S_a and ν_a also must be determined. This makes it very difficult to solve this system of equations even for a two-component plasma. The problem may be greatly simplified by using a model collision integral.

We calculate the electron distribution function in the one-dimensional case on the assumption that a longitudinal wave distorts f_e only in its direction of propagation along $\mathbf{k}$. It is convenient to take the expression representing S_e in the following form for Coulomb collisions:

$$S_{ei} = \frac{3}{2}\frac{1}{\tau_{ei}^{(T)}}\frac{d}{dv}\left\{\frac{T_i}{m}\frac{df_e}{dv} + v f_e\right\}\,, \tag{3.1}$$

in which T_i is ion temperature, while $\tau_{ei}^{(T)}$ defines the relaxation time of T_i and T_e.

Integral S_e in the form of (3.1) takes into account only electron-ion collisions, since the term corresponding to electron-electron collisions is nonlinear. Ginzburg [7] has shown that these collisions need be considered only for low frequencies, when $\omega^2 \ll \nu_*^2$, in which ν_* is some effective collisional frequency that defines the contribution to the conductivity. We can use a model collision integral in the Bathnagar-Gross-Crook form in order to take into account the collisions of electrons with molecules in a weakly ionized plasma, but this gives a more complicated differential equation for f_e.

The integral of (3.1) provides conservation of momentum, energy, and number of particles, and also $S_{ei} = 0$ in a state of equilibrium, when f_e is Maxwellian and $T_e = T_i$.

Removing one differentiation, we have from (0.1) with (3.1) the equation for f_e:

$$\left\{1 + \frac{\delta^3 l}{(\omega - kv)^2 + \mathrm{v}_e^2}\right\}\frac{df_e}{dv} + \frac{mv}{T_i} f_e = 0 \tag{3.2}$$

in which

$$\delta = \left(\frac{e^2 |E|^2 \mathrm{v}_e \tau_{ei}^{(T)}}{3mT_i}\right)^{1/3}, \tag{3.3}$$

has the dimensions of frequency and may be considered as the effective frequency of the collisions of the electrons with the wave.

It is simple to solve the linear homogeneous equation (3.2); we put $v_* = (\omega + i\alpha)/k$ and $\alpha^2 = \delta^2 + \nu_e^2$ to express the solution as

$$f_e = C\exp\left\{-\frac{mv^2}{2T_i} + \frac{m\delta^2}{T_i \alpha k}\,\mathrm{Im}\left[v_* \ln\left(1 - \frac{v}{v_*}\right)\right]\right\}, \tag{3.4}$$

in which C is a constant of integration.

The first term in the exponential corresponds to a Maxwellian distribution with a temperature equal to T_i, while the second is a correction dependent on the wave amplitude. It is clear from (3.4) that the field correction is maximum for velocities near ω/k, the phase velocity of the wave. If we completely neglect spatial dispersion and expand the logarithm as a power series in $1/v_*$, we get (3.4) as

$$f_e = C \exp\left\{-\frac{mv^2}{2T_i(1+\delta^2/(\omega^2+\nu_e^2))}\right\}. \qquad (3.5)$$

Then the electron distribution function is exactly Maxwellian for $k = 0$, but with an increased temperature, which is defined by (1.12), as is readily shown from (3.3).

The figure shows curves for the electron distribution function in the quasi-linear approximation, as calculated from (3.4) and (3.5). The abscissa is the dimensionless velocity $\zeta = kv/\omega$, while the ordinate is the dimensionless function $F(\zeta)$, which equals the square root of the exponent divided by $m\omega^2/2T_ik^2$. The curves are for different values of δ/ω, which characterizes the relative energy of the wave field: 1 and 2 correspond to $\delta/\omega = 0.1$, and 3 and 4 to $\delta/\omega = 1$. It was also assumed that $\delta^2 \gg \nu_e^2$ in the calculations. Curves 1 and 3 correspond to a Maxwellian distribution with an increased temperature [formula (3.5)] and are straight lines, while curves 2 and 4 show the occurrence of a plateau on the distribution function near the phase velocity of the wave [formula (3.4)]. It is clear that the width of the plateau increases with the amplitude of the wave. If we expand the exponent in (3.4) as a power series in $(v - \omega/k)$ and retain terms only up to the second degree, we can show that the distribution function in the plateau region is also close to Maxwellian, but with a different effective temperature:

$$T_* = T_i(1+\delta^2/\nu_e^2).$$

The plateau becomes strictly horizontal ($\nu_e \to 0$, $T_* \to \infty$) in a plasma without collisions.

REFERENCES

1. Yu. L. Klimontovich, Statistical Theory of Nonequilibrium Plasma Processes [in Russian], Izd. MGU, 1964 (see also the supplement to the English edition).
2. V. P. Silin, "High-frequency dielectric constant of a plasma," ZhETF, 41, 861, 1961.
3. V. V. Loginov, "Relaxation in a Coulomb quasi-equilibrium plasma," Vestnik Mosk. un-ta., no. 1, 1967.
4. V. L. Ginzburg and A. V. Gurevich, "Nonlinear phenomena in a plasma in an alternating electromagnetic field," Usp. fiz. nauk, 70, no. 2, 1960.
5. D. V. Sivukhin, "Coulomb collisions in a completely ionized plasma," collection: Plasma Theory, No. 4 [in Russian], Atomizdat, 1964.
6. Ya. B. Fainberg and V. D. Shapiro, "Quasi-linear theory of wave excitation in the injection of an electron beam into a plasma half-space," ZhETF, 47, 1389, 1964.
7. V. L. Ginzburg, Propagation of Electromagnetic Waves in a Plasma [in Russian], Fizmatgiz, 1960.

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